GENERALIZED GRAMIANS: CREATING FRAME VECTORS IN MAXIMAL SUBSPACES

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ABSTRACT. A frame is a system of vectors S in Hilbert space $\mathscr{H}$ with properties which allow one to write algorithms for the two operations, analysis and synthesis, relative to S, for all vectors in $\mathscr{H}$; expressed in norm-convergent series. Traditionally, frame properties are expressed in terms of an S-Gramian, G_S (an infinite matrix with entries equal to the inner product of pairs of vectors in S); but still with strong restrictions on the given system of vectors in S, in order to guarantee frame-bounds. In this paper we remove these restrictions on G_S, and we obtain instead direct-integral analysis/synthesis formulas. We show that, in spectral subspaces of every finite interval J in the positive half-line, there are associated standard frames, with frame-bounds equal the endpoints of J. Applications are given to reproducing kernel Hilbert spaces, and to random fields.

CONTENTS

1. INTRODUCTION

Frames are redundant bases which turn out in certain applications to be more flexible than the better known orthonormal bases (ONBs) in Hilbert space; see, e.g., [BCL11,

Chr14, HKLW07, FJKO05a]. The frames allow for more symmetries than ONBs do, especially in the context of signal analysis, and of wavelet constructions; see, e.g., [CD93, Bil05, Dut06]. Since frame bases (although containing redundancies) still allow for efficient algorithms, they have found many applications, even in finite dimensions; see, for example, [BF03, CC03, Chr03, Eld02, FJKO05b].

As is well known, when a vector f in a Hilbert space $\mathscr{H}$ is expanded in an orthonormal basis B, there is then automatically an associated Parseval identity. In physical terms, this identity typically reflects a *stability* feature of a decomposition based on the chosen ONB B. Specifically, Parseval's identity reflects a conserved quantity for a problem at hand, for example, energy conservation in quantum mechanics.

The theory of frames begins with the observation that there are useful vector systems which are in fact not ONBs but for which a Parseval formula still holds. In fact, in applications it is important to go beyond ONBs. While this viewpoint originated in signal processing (in connection with frequency bands, aliasing, and filters), the subject of frames appears now to be of independent interest in mathematics.

Recently the solution to the Kadison-Singer question, and its equivalent forms, has proved to take a form involving frames in Hilbert space. While we will not discuss the Kadison-Singer question here, we mention it as an illustration of the usefulness of frames in problems which in their original form were not stated in the language of frames. The Kadison-Singer question is about uniqueness of extensions of pure states, where a pure state is a quantum theoretical notion. For interested readers, we refer to [AW14, Cas13, Cas14, MSS13].

This entails systems of vectors $\{\varphi_n\}_{n \in \mathbb{N}} \subset \mathscr{H}$, called *frames*, where there are two constants a, b, s.t. $0 < a \leq b < \infty$, satisfying:

$$a \left\| f \right\|_{\mathscr{H}}^2 \leq \sum_{n=1}^{\infty} \left| \langle \varphi_n, f \rangle_{\mathscr{H}} \right|^2 \leq b \left\| f \right\|_{\mathscr{H}}^2, \quad \forall f \in \mathscr{H}. \tag{1.1}$$

The two constants a and b in (1.1) are called *frame bounds*. (*Parseval frame* means $a = b = 1$.)

The purpose of this paper is to associate canonical *projection valued measures* (PVM) to system of vectors $\{\varphi_n\}$ which are not frames but such that

$$\sum_{j \in \mathbb{N}} \left| \langle \varphi_j, \varphi_n \rangle_{\mathscr{H}} \right|^2 < \infty, \quad \forall n \in \mathbb{N}; \tag{1.2}$$

and, from this, then to generate systems of *frames* in suitable canonical closed subspaces of $\mathscr{H}$, with the subspaces derived from the PVM, and with a variety of choices of *frame bounds*.

As an application, we give natural and explicit conditions on these frame systems to generate reproducing kernel Hilbert spaces (RKHS).

The RKHSs have been studied extensively since the pioneering papers by Aronszajn in the 1940ties, see e.g., [Aro43, Aro48]. They further play an important role in the theory of partial differential operators (PDO); for example as Green's functions of second order elliptic PDOs; see e.g., [Nel57, HKL$^+$14]. Other applications include engineering, physics, machine-learning theory (see [KH11, SZ09, CS02]), stochastic processes (e.g., Gaussian free fields), numerical analysis, and more. See, e.g., [AD93, ABDdS93, AD92, AJSV13, AJV14], and also [ZXZ12, LP11, Vul13, SS13].

2. Gramians

Let $\mathscr{H}$ be a Hilbert space, and $\{\varphi_n\}_{n\in\mathbb{N}}$ an indexed system of vectors in $\mathscr{H}$; then we say that the $\infty \times \infty$ matrix

$$G = \left(\langle \varphi_i, \varphi_j \rangle_{\mathscr{H}}\right)_{i,j=1}^{\infty} \tag{2.1}$$

is a *Gramian*. Gramians are so named since they were first used in the Gram-Schmidt orthogonalization; but, more recently, they have played an important role in the analysis of frames. In order to get frames, it is natural to further assume that

$$\{\varphi_n\}_{n\in\mathbb{N}}^{\perp} = \{0\}, \tag{2.2}$$

i.e., that the vectors span a dense subspace in $\mathscr{H}$.

In order to understand frame properties indexed by such systems, it is necessary to interpret G in (2.1) as an operator in l^2. This works if we further assume:

$$\text{For } \forall n \in \mathbb{N}, \quad \sum_{j\in\mathbb{N}} \left|\langle \varphi_j, \varphi_n \rangle_{\mathscr{H}}\right|^2 < \infty. \tag{2.3}$$

In that case, there is an operator T_G as follows:

$$(T_G c)_n = \sum_{j=1}^{\infty} \langle \varphi_n, \varphi_j \rangle_{\mathscr{H}} c_j, \quad \forall c = (c_j) \in l^2, \tag{2.4}$$

or c in a dense subspace of l^2. More precisely, T_G defines a Hermitian semi-bounded operator $T_G : l^2 \to l^2$ with dense domain (finite sequences) in l^2. But in general, T_G

(from (2.4)) is unbounded, its domain is only dense in l^2; and it may not even be essentially selfadjoint.

By von Neumann's theory [DS88], T_G is essentially selfadjoint if and only if the following holds:

$$\left\{ \xi \in l^2 \,\middle|\, T_G^* \xi = -\xi \right\} = 0. \tag{2.5}$$

One checks that (2.5) holds precisely when

$$\left[\sum_{j=1}^{\infty} \langle \varphi_j, \varphi_n \rangle_{\mathscr{H}} \, \xi_j = -\xi_n \right] \Longrightarrow \xi = 0 \text{ in } l^2. \tag{2.6}$$

Using a theorem from [Jør77, Jør78], one further checks that (2.5), or equivalently, (2.6), holds if the Gramian satisfies:

$\exists \{b_n\} \subset \mathbb{R}_+$ such that

$$\sum_{j \neq n} \left| \langle \varphi_j, \varphi_n \rangle_{\mathscr{H}} \right| = O(b_n), \text{ and}$$

$$\sum_n \frac{1}{\sqrt{b_n}} = \infty. \tag{2.7}$$

Remark 2.1. Using Aronszajn's theorem, one shows that the following two $(1)\&(2)$ are the "same":

(1) $p : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C}$ positive definite function; and
(2) a system $(\mathscr{H}, \{\varphi_i\}_{i \in \mathbb{Z}_+} \subset \mathscr{H})$ s.t. $\{\varphi_i\}^{\perp} = 0$, and

$$p(i,j) = \langle \varphi_i, \varphi_j \rangle_{\mathscr{H}}, \quad (i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+. \tag{2.8}$$

Let $\mathscr{F} \subset l^2$ denote the (dense) subspace of all *finite* sequence: If

$$\sum_j |p(i,j)|^2 = \sum_j |\langle \varphi_i, \varphi_j \rangle|^2 < \infty, \quad \forall i \in \mathbb{Z}_+, \tag{2.9}$$

then we get an operator T_p in l^2 as follows:

$$(T_p c)_i := \sum_j p(i,j) c_j, \quad \forall c \in \mathscr{F} \tag{2.10}$$

defines a Hermitian semibounded operator (generally unbounded) in l^2 with dense domain $dom(T_p) = \mathscr{F}$, i.e.,

$$\langle c, T_p c \rangle_{l^2} \geq 0, \quad \forall c \in \mathscr{F}. \tag{2.11}$$

3. Main theorems

Definition 3.1. Let $\mathscr{H}$ be a Hilbert space, and let $\{\varphi_n\}_{n=1}^{\infty} \subset \mathscr{H}$ be a system of vectors s.t. $\{\varphi_n\}^{\perp} = 0$, i.e., the span is dense in $\mathscr{H}$. Let $a, b \in \mathbb{R}_+$, be given such that $0 < a \leq b < \infty$; and let $\mathscr{K} \subset \mathscr{H}$ be a closed subspace. We say that $\mathscr{K}$ is an (a, b) *frame subspace* iff (Def)

$$a \|f\|^2 \leq \sum_{n=1}^{\infty} |\langle \varphi_n, f \rangle|^2 \leq b \|f\|^2, \tag{3.1}$$

holds for all vectors $f \in \mathscr{K}$. Here $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathscr{H}$, and $\|f\| = \langle f, f \rangle^{\frac{1}{2}}$ the corresponding norm.

Theorem 3.2. *Let $\mathscr{H}$, $\{\varphi_n\}$ be a pair, $\mathscr{H}$ a Hilbert space, and $\{\varphi_n\}_1^{\infty} \subset \mathscr{H}$, satisfying $\{\varphi_n\}^{\perp} = 0$. Assume further that*

$$\sum_{j=1}^{\infty} |\langle \varphi_n, \varphi_j \rangle|^2 < \infty, \quad \forall n \in \mathbb{N}, \ and \tag{3.2}$$

$$\left[\xi \in l^2, \ \sum_j \langle \varphi_n, \varphi_j \rangle \xi_j = -\xi_n \right] \Longrightarrow \xi = 0 \ in \ l^2. \tag{3.3}$$

Then, for every $a, b \in \mathbb{R}_+$, s.t. $0 < a \leq b < \infty$, there is a unique <u>maximal</u> subspace $\mathscr{K} = \mathscr{H}(a, b)$ which satisfies the frame condition (3.1).

Proof. We begin with a specification of a pair of dense subspaces in $\mathscr{H}$, and in l^2, $\mathscr{D}_{\mathscr{H}}$, and $\mathscr{D}_{l^2}$:

> $\mathscr{D}_{\mathscr{H}} = span\{\varphi_n\}_{n \in \mathbb{N}}$, i.e., all finite linear combination of these vectors, and
> $\mathscr{D}_{l^2} = span\{\varepsilon_n\}$, where $\varepsilon_n(j) = \delta_{nj}$; i.e., $\mathscr{D}_{l^2}$ consists of all finitely supported sequences.

We introduce two densely defined operators

$$\mathscr{H} \xrightarrow{L} l^2, \quad \text{and} \quad l^2 \xrightarrow{M} \mathscr{H} \tag{3.4}$$

as follows: Set the respective domains,

$$dom(L) = \mathscr{D}_{\mathscr{H}}, \quad \text{and} \quad dom(M) = \mathscr{D}_{l^2}, \tag{3.5}$$

and set

$$Lf = (\langle \varphi_n, f \rangle)_n, \quad \forall f \in \mathscr{D}_{\mathscr{H}}; \tag{3.6}$$

$$M\xi = \sum_{n=1}^{\infty} \xi_n \varphi_n, \quad \forall \xi \in \mathscr{D}_{l^2}. \tag{3.7}$$

From the definition of these two operators, we get the following symmetry:

$$\langle Lf, \xi \rangle_{l^2} = \langle f, M\xi \rangle_{\mathscr{H}}, \quad \forall f \in \mathscr{D}_{\mathscr{H}}, \forall \xi \in \mathscr{D}_{l^2}, \tag{3.8}$$

where the respective inner products are indicated with subscripts, $\langle \cdot, \cdot \rangle_{\mathscr{H}}$ and $\langle \cdot, \cdot \rangle_{l^2}$.

If L^* and M^* denote the respective adjoint operators, then

$$l^2 \xrightarrow{L^*} \mathscr{H}, \quad \text{and} \quad \mathscr{H} \xrightarrow{M^*} l^2. \tag{3.9}$$

We will need the following: □

Lemma 3.3. *We have*

$$L \subseteq M^*, \quad and \quad M \subseteq L^*, \tag{3.10}$$

where the containments in (3.10) refer to the respective graphs, i.e., (3.10) states that

$$G(L) \subseteq G(M^*), \quad and \quad G(M) \subseteq G(L^*). \tag{3.11}$$

By the graph of a linear operator $T : \mathscr{H}_1 \to \mathscr{H}_2$ between Hilbert spaces, we mean

$$G(T) = \left\{ \begin{pmatrix} h_1 \\ Th_1 \end{pmatrix} \; \middle| \; h_1 \in dom(T) \right\} \subset \begin{pmatrix} \mathscr{H}_1 \\ \oplus \\ \mathscr{H}_1 \end{pmatrix}. \tag{3.12}$$

Proof. The conclusion in the lemma is immediate from (3.8). □

Corollary 3.4. *Both of the operators L and M in (3.6)-(3.7) are <u>closable</u>, i.e., the closure of $G(L)$ is a densely defined closed operator $\mathscr{H} \xrightarrow{\overline{L}} l^2$, and the closure of $G(M)$ is a densely defined closed operator $l^2 \xrightarrow{\overline{M}} \mathscr{H}$.*

Proof. Follows from the fact that a densely defined operator $\mathscr{H}_1 \xrightarrow{T} \mathscr{H}_2$ is closable if and only if the domain of the adjoint T^* is dense in $\mathscr{H}_2$. The assertion that both L^* and M^* have dense domains is immediate from (3.10). □

Lemma 3.5. *Let L and M be the two operators in Lemma 3.3, then*

$$\overline{L} = M^*, \quad and \quad \overline{M} = L^*. \tag{3.13}$$

Proof. From (3.10), we automatically get containments:

$$\overline{L} \subseteq M^*, \quad and \quad \overline{M} \subseteq L^*, \tag{3.14}$$

but the conclusion (3.13) in the lemma, is that both containments (3.14) are equality.

Returning to the definitions (3.6) and (3.7), we note that (3.13) follows if we prove the following two implications:

$$\left\{ \begin{array}{l} f \in dom(M^*), \text{ and} \\ \langle \varphi_n, f \rangle_{\mathscr{H}} + \sum_{j=1}^{\infty} \langle \varphi_n, \varphi_j \rangle_{\mathscr{H}} (M^* f)_j = 0 \end{array} \right\} \implies f = 0 \text{ in } \mathscr{H}. \qquad (3.15)$$

And

$$\left\{ \begin{array}{l} \xi \in dom(L^*), \text{ and} \\ \sum_{j=1}^{n} \langle \varphi_n, \varphi_j \rangle_{\mathscr{H}} \xi_j = -\xi_n, \ \forall n \in \mathbb{N} \end{array} \right\} \implies \xi = 0 \text{ in } l^2. \qquad (3.16)$$

Using the symmetry (3.14), we note that either one of the two implications (3.15), or (3.16), implies the other.

We will prove (3.16). For this, we make use of assumption (3.2) and (3.3) from the premise in Theorem 3.2.

Here the $\infty \times \infty$ matrix $\left(\langle \varphi_n, \varphi_j \rangle_{\mathscr{H}} \right)$ defines a Hermitian symmetric operator T with dense domain in l^2,

$$(Tc)_n := \sum_{j} \langle \varphi_n, \varphi_j \rangle_{\mathscr{H}} c_j, \quad \forall c = (c_j) \in dom(T). \qquad (3.17)$$

Since T is semibounded, i.e.,

$$\langle c, Tc \rangle_{l^2} \geq 0, \quad \forall c \in dom(T), \qquad (3.18)$$

it follows from assumption (3.3) that T has deficiency indices $(0,0)$; and in particular that the implication (3.16) holds.

This completes the proof of Lemma 3.5. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Theorem 3.2, continued. We conclude from the lemmas that $L_1 := \overline{L}$ (closure) is a closed operator with dense domain in $\mathscr{H}$, and $\mathscr{H} \xrightarrow{\overline{L}} l^2$.

By a theorem of von Neumann [DS88], we further conclude that the two operators $L^* L_1$, and $L_1 L^*$, are both selfadjoint (s.a.); the first in $\mathscr{H}$, and the second in l^2 (not merely Hermitian symmetric), but both have deficiency indices $(0,0)$, and both are semibounded, i.e., with spectrum in $[0, \infty]$.

Since $L^* L_1$ is s.a. with dense domain in $\mathscr{H}$, it has a unique spectral resolution in the form of a projection valued measure $P(\cdot)$, i.e.,

(1) $P(A) = P(A)^* = P(A)^2$, $\forall A \in \mathscr{B} := \mathscr{B}([0, \infty))$;
(2) $P(A \cap B) = P(A) P(B)$, $A, B \in \mathscr{B}$;

(3) $P(\cdot)$ is sigma-additive;

(4) $P([0, \infty)) = I_{\mathscr{H}}$, the identity operator in $\mathscr{H}$; and

(5) $L^* L_1 = \int_0^\infty \lambda P(d\lambda)$.

More generally, if φ is a Borel function, then by functional calculus the operator $\varphi(L^* L_1)$ is:

$$\varphi(L^* L_1) = \int_0^\infty \varphi(\lambda) P(d\lambda); \tag{3.19}$$

and the following are equivalent:

$$f \in dom(\varphi(L^* L_1)) \Longleftrightarrow \int_0^\infty |\varphi(\lambda)|^2 \|P(d\lambda) f\|_{\mathscr{H}}^2 < \infty. \tag{3.20}$$

If $\alpha = [a, b]$, $0 < a \le b < \infty$, i.e., a fixed compact interval in $(0, \infty)$, set

$$\mathscr{H}_\alpha = P(\alpha) \mathscr{H}. \tag{3.21}$$

Setting function $\varphi(\lambda) = \lambda \chi_\alpha(\lambda)$ in (3.19), we get

$$a \|f\|_{\mathscr{H}}^2 \le \langle f, L^* L_1 f \rangle_{\mathscr{H}} \le b \|f\|_{\mathscr{H}}^2, \quad \forall f \in \mathscr{H}_\alpha. \tag{3.22}$$

Since $\langle f, L^* L_1 f \rangle_{\mathscr{H}} = \sum_{n=1}^\infty |\langle \varphi_n, f \rangle_{\mathscr{H}}|^2$, it follows that (3.22) is the desired frame-estimate, i.e.,

$$a \|f\|_{\mathscr{H}}^2 \le \sum_{n=1}^\infty |\langle \varphi_n, f \rangle_{\mathscr{H}}|^2 \le b \|f\|_{\mathscr{H}}^2, \quad \forall f \in \mathscr{H}_\alpha. \tag{3.23}$$

Moreover, if $\mathscr{K} \subset \mathscr{H}$ is a closed subspace in $\mathscr{H}$ s.t. (3.23) holds for $\forall f \in \mathscr{K}$, then it follows from (3.19)-(3.21) that $\mathscr{K} \subset \mathscr{H}_\alpha$, $\alpha = [a, b]$. This concludes the proof of the theorem. $\qquad \square$

The following result is implied by Lemma 3.5, and is of independent interest:

Proposition 3.6. *Let $\left(\mathscr{H}, \{\varphi_n\}_{n \in \mathbb{N}}\right)$ satisfy the conditions (3.2)-(3.3) in Theorem 3.2, and let*

$$L_1 = U(L^* L_1)^{\frac{1}{2}} = (L_1 L^*)^{\frac{1}{2}} U \tag{3.24}$$

be the corresponding polar decomposition of $\mathscr{H} \xrightarrow{L_1} l^2$, then

$$U = T_G^{-\frac{1}{2}} L_1, \ and \tag{3.25}$$

U is an isometry of $\mathscr{H}$ into l^2.

Proof. By the properties of the polar decomposition (3.24), U is a partial isometry with initial space $= \mathscr{H} \ominus \ker(L_1)$; so we only need to prove that $\ker(L_1) = 0$. But from (3.6) we get $\ker(L_1) = \{\varphi_n\}^\perp = 0$, by (2.2). $\qquad \square$

Definition 3.7. Let Ω be a set, and let $\mathscr{H}$ be a Hilbert space of functions on Ω. We say that $\mathscr{H}$ is a *reproducing kernel Hilbert* space (RKHS) iff (Def): $\mathscr{H} \ni f \longmapsto f(t) \in \mathbb{C}$ is norm-continuous for all $t \in \Omega$; or, equivalent, if for all $t \in \Omega$, there exists a $K_t \in \mathscr{H}$ s.t.

$$f(t) = \langle K_t, f \rangle_{\mathscr{H}}, \quad \forall f \in \mathscr{H}. \tag{3.26}$$

We say that (3.26) is the reproducing property. (See, e.g., [Aro43, Aro48].)

Theorem 3.8. *Let $\mathscr{H}$ be a Hilbert space, and $\{\varphi_n\}_{n \in \mathbb{N}}$ be a <u>frame</u> with frame bounds a, b, where*

$$0 < a \leq b < \infty. \tag{3.27}$$

Suppose further that there is a set Ω s.t. each φ_n is a function on Ω, and that

$$(\varphi_n(t))_{n \in \mathbb{N}} \in l^2, \quad \forall t \in \Omega. \tag{3.28}$$

Then $\mathscr{H}$ is a reproducing kernel Hilbert space (RKHS) of functions on Ω.

Proof. By (3.27)&(3.28) we conclude that

$$\sum_{n \in \mathbb{N}} \varphi_n(t)\, \varphi_n \in \mathscr{H}, \tag{3.29}$$

and that

$$a \sum_n |\varphi_n(t)|^2 \leq \left\| \sum_{n=1}^{\infty} \varphi_n(t)\, \varphi_n \right\|_{\mathscr{H}}^2 \leq b \sum_n |\varphi_n(t)|^2 \tag{3.30}$$

holds for all $t \in \Omega$.

For $t \in \Omega$, set

$$l(t) = \begin{bmatrix} \varphi_1(t) \\ \varphi_2(t) \\ \vdots \\ \vdots \end{bmatrix} \in l^2 \tag{3.31}$$

and set

$$\begin{aligned} K^G(s,t) &:= \langle l(s), G^{-1} l(t) \rangle_2 \tag{3.32} \\ &= \sum \sum_{(n,m) \in \mathbb{N} \times \mathbb{N}} \overline{\varphi_n(s)}\, (G^{-1})_{nm}\, \varphi_m(t). \end{aligned}$$

We claim that $K^G(\cdot, \cdot)$ in (3.32) turns $\mathscr{H}$ into a RKHS.

To see this, set

$$Lf = (\langle \varphi_j, f \rangle_{\mathscr{H}})_{j \in \mathbb{N}}, \text{ and} \tag{3.33}$$

$$L^*c \;=\; \sum_{j=1}^{\infty} c_j \varphi_j \tag{3.34}$$

then by standard frame theory L^*L and LL^* are both bounded and selfadjoint. Moreover, $LL^* = T_G =$ the operator in l^2 defined from the Gramian matrix

$$G = \left(\langle \varphi_i, \varphi_j \rangle_{\mathscr{H}} \right)_{i,j=1}^{\infty}. \tag{3.35}$$

Since a, b are the frame bounds, we get:

$$aI_{\mathscr{H}} \le L^*L \le bI_{\mathscr{H}}, \text{ and} \tag{3.36}$$

$$aI_{l^2} \le LL^* \le bI_{l^2} \tag{3.37}$$

where I denotes the respective identity operators in $\mathscr{H}$, and in l^2; and where "$\le$" denotes the natural ordering of Hermitian (bounded) operators. It further follows that the system,

$$\psi_n := (L^*L)^{-\frac{1}{2}} \varphi_n, \quad n \in \mathbb{N} \tag{3.38}$$

is a *Parseval* frame in $\mathscr{H}$, i.e.,

$$\|f\|_{\mathscr{H}}^2 = \sum_{n=1}^{\infty} |\langle \psi_n, f \rangle_{\mathscr{H}}|^2 \tag{3.39}$$

holds for all $f \in \mathscr{H}$.

Using (3.29), we see that every $f \in \mathscr{H}$ identifies with a function on Ω; and we shall use the same notation for this function representations. By (3.39),

$$f = \sum_{n=1}^{\infty} \langle \psi_n, f \rangle_{\mathscr{H}} \, \psi_n \tag{3.40}$$

holds for $f \in \mathscr{H}$, with norm-convergence. Hence (using the function-representation):

$$
\begin{aligned}
f(t) \;&=\; \sum_{n=1}^{\infty} \langle \psi_n, f \rangle_{\mathscr{H}} \, \psi_n(t) \\
&=\; \left\langle \sum_{n=1}^{\infty} \overline{\psi_n(\cdot)} \psi_n, f \right\rangle_{\mathscr{H}} \\
&=\; \left\langle \sum_{n=1}^{\infty} \overline{\left((L^*L)^{-\frac{1}{2}} \varphi_n \right)}(\cdot) \left((L^*L)^{-\frac{1}{2}} \varphi_n \right)(t), f \right\rangle_{\mathscr{H}} \\
&=\; \left\langle \left\langle T_G^{-\frac{1}{2}} l(\cdot), T_G^{-\frac{1}{2}} l(t) \right\rangle_2, f \right\rangle_{\mathscr{H}}
\end{aligned}
$$

$$= \ \left\langle K^G\left(\cdot,t\right),f\right\rangle_{\mathscr{H}},$$

valid for $\forall t \in \Omega$, and all $f \in \mathscr{H}$. This is the desired RKHS-property for $\mathscr{H}$. $\qquad\square$

4. APPLICATIONS

4.1. Hilbert matrix

In Theorem 3.8 we assume that the given Hilbert space $\mathscr{H}$ has a frame $\{\varphi_n\} \subset \mathscr{K}$ consisting of functions on a set Ω. So this entails both a lower, and an upper bound, i.e., $0 < B_1 \leq B_2 < \infty$. The following example shows that the conclusion in the theorem is false if there is *not* a positive lower frame bound.

Set $\mathscr{H} = L^2\left(0,1\right)$, $\Omega = (0,1)$, the open unit-interval, and $\varphi_n\left(t\right) = t^n$, $n \in \{0\}\cup\mathbb{N} = \mathbb{N}_0$. In this case, the Gramian

$$G_{n,m} = \int_0^1 x^{n+m}dx = \frac{1}{n+m+1} \tag{4.1}$$

is the $\infty \times \infty$ Hilbert matrix, see [Ros58, Kat57, Tau49]. In this case it is known that there is an upper frame bound $B_2 = \pi$, i.e.,

$$\sum_{n=0}^{\infty}\left|\int_0^1 f\left(x\right)x^n dx\right|^2 \leq \pi\int_0^1\left|f\left(x\right)\right|^2 dx; \tag{4.2}$$

in fact, for the operator-norm, we have $\|G\|_{l^2\to l^2} = \pi$.

Moreover, G defines a selfadjoint operator in $l^2\left(\mathbb{N}_0\right)$ with spectrum $[0,\pi]$, the closed interval. This implies that there cannot be a positive lower frame-bound.

Further, it is immediate by inspection that $\mathscr{H} = L^2\left(0,1\right)$ is *not* a RKHS.

4.2. Random fields

Definition 4.1 (see [AY09]). Let V be a countable discrete set (for example the vertex-set in an infinite graph [Hua14, GI13].) Let $(\Omega, \mathscr{F}, P)$ be a probability space, and let $\mathscr{H} := L^2\left(\Omega, P\right)$ be the corresponding Hilbert space of L^2-random variables, i.e., $f : \Omega \to \mathbb{R}$, measurable s.t.

$$\mathbb{E}(|f|^2) = \int_\Omega |f|^2\,dP < \infty. \tag{4.3}$$

A system $(\varphi_x)_{x\in V} \subset \mathscr{H}$ is said to be a *random frame* if

$$\sum_{y\in V}\left|\mathbb{E}\left(\varphi_x\varphi_y\right)\right|^2 < \infty, \quad \forall x \in V; \tag{4.4}$$

and the following implication holds:

$$\left[\sum_{y \in V} \mathbb{E}\left(\varphi_x \varphi_y\right) c_y = -c_x, \ (c_x) \in l^2\left(V\right)\right] \Longrightarrow (c_x) = 0 \text{ in } l^2\left(V\right). \qquad (4.5)$$

Corollary 4.2. *Let* $(\Omega, \mathscr{F}, P)$ *be a probability space and let* $(\varphi_x)_{x \in V}$ *be a random frame in* $L^2\left(\Omega, P\right)$. *Then*

(1) *the* <u>*generalized frame operator*</u>

$$L^* L_1 f = \sum_{x \in V} \mathbb{E}\left(\varphi_x f\right) \varphi_x \qquad (4.6)$$

is selfadjoint in $L^2\left(\Omega, P\right)$, *generally unbounded; and*

(2) *Let* $a, b \in \mathbb{R}_+$ *satisfying* $0 < a \leq b < \infty$, *set* $\alpha = [a, b]$. *Let* $P_E\left(\cdot\right)$ *be the projection valued measure of* $L^* L_1$, *and* $\mathscr{H}\left(\alpha\right) = P_E\left(\alpha\right) \mathscr{H}$, *then the estimate*

$$a \, \mathbb{E}(|f|^2) \leq \sum_{x \in V} \left|\mathbb{E}\left(\varphi_x f\right)\right|^2 \leq b \, \mathbb{E}(|f|^2) \qquad (4.7)$$

holds for all $f \in \mathscr{H}\left(\alpha\right)$. *(i.e., If* $P_E\left(\cdot\right)$ *denotes the spectral resolution of the operator* $L^* L_1$ *in* (1), *we may take* $\mathscr{H}\left(\alpha\right) = P_E\left(\alpha\right) \mathscr{H} \left(= P_E\left([a, b]\right) \mathscr{H}\right)$ *in* (4.7).)

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